Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(00)00087-3
J. Appl. Maths Mechs, Vol. 64, No. 4, pp. 607–617, 2000

0021-8928/00/$—see front matter

THE TAYLOR AND HYPERBOLIC MODELS OF UNSTEADY LONGITUDINAL DISPERSION OF A PASSIVE IMPURITY IN CONVECTION-DIFFUSION PROCESSES†

A. D. KHON'KIN

Zhukovskii

(*Received* 12 *August* 1999)

The problem of the dispersion of a passive impurity in a circular tube is considered. Using an asymptotic method, similar to the Chapman–Enskog method of the kinetic theory of gases or the Krylov–Bogoloyubov averaging method of non-linear mechanics, an equation of steady diffusion is derived for the impurity concentration, averaged over the tube cross-section, which was obtained by Taylor from physical considerations. Using the asymptotic method a recurrence system of equations is obtained for the expansion terms of arbitrary order. Estimates are made of the applicability of Taylor's model of longitudinal dispersion, which refines the estimated established by Taylor. To extend the limits of applicability of Taylor's model a two-term Bubnov–Galerkin representation is employed for the concentration, averaged over the tube cross-section, which is now described by a hyperbolic-type telegraph equation. Green's function for this model is obtained, according to which the impurity concentration distribution is characterized by the presence of perturbation fronts with finite propagation velocities. The asymptotic agreement between Green's function of the hyperbolic model and Green's function of the Taylor model is demonstrated. © 2000 Elsevier Science Ltd. All rights reserved.

When investigating the problem of the longitudinal dispersion of a passive impurity in a fluid flow in a circular tube, Taylor [1, 2] found that the interaction of convection and transverse diffusion processes generates a specific mechanism of longitudinal dispersion, which is much stronger than the molecular diffusion mechanism. The equation Taylor obtained for the impurity concentration, averaged over the tube cross-section, has found wide application when constructing one-dimensional models of different devices such as, for example, circulating thermal and chemical reactors. However, Taylor's approach is based on physical considerations, so that there is a need to derive Taylor's equation using some regular procedure of successive approximations. Moreover, Taylor's model leads to a parabolic equation of unsteady diffusion and, consequently, to the well-known paradox of an infinite perturbation propagation velocity.

We show below that Taylor's model can be obtained using an asymptotic method, in which a small parameter and estimates of the applicability of the model are determined, and a hyperbolic model of longitudinal dispersion with finite perturbation propagation velocities is obtained, which is asymptotically equivalent to Taylor's model in regions far from the perturbation fronts.

1. THE ASYMPTOTIC THEORY OF TAYLOR DISPERSION OF A PASSIVE IMPURITY

In the simplest case of a linear mechanism of unsteady dispersion of a passive impurity, when there are no physical–chemical changes, the problem is described by the unsteady convection-diffusion equation for a passive impurity concentration $c(\mathbf{R}, t)$

$$\frac{\partial c}{\partial t} + (\mathbf{V} \cdot \nabla)c = (\nabla \cdot \mathbf{D} \cdot \nabla)c \tag{1.1}$$

Here $\mathbf{V}$ is the velocity vector of the carrying medium (a liquid or a gas), $\mathbf{R}$ is the radius vector of a point in space and $\mathbf{D}$ is the tensor of the diffusion coefficients. In the general case both $\mathbf{V}$ and $\mathbf{D}$ may be functions of time and the coordinates, which are assumed known.

Taylor considered the axisymmetric problem of unsteady dispersion of a passive impurity in a cylindrical tube of radius a, described by the equation

†*Prikl. Mat. Mekh.* Vol. 64, No. 4, pp. 631–643, 2000.

$$\frac{\partial c}{\partial t}+u(r)\frac{\partial c}{\partial x}=\frac{1}{r}\frac{\partial}{\partial r}\left[D(r)r\frac{\partial c}{\partial r}\right] \tag{1.2}$$

and the boundary condition

$$\partial c/\partial r=0 \quad \text{when} \quad r=a \tag{1.3}$$

where x is the longitudinal coordinate, measured along the tube axis, r is the radial coordinate, measured from the tube axis, $u(r)$ is the Poiseuille laminar velocity profile [1] or the turbulent velocity profile, which was specified in (2) by a table of experimental values, and $D(r)$ is the molecular or turbulent diffusion coefficient of the impurity. When writing the convection-diffusion equation in the form (1.2) Taylor assumed that the effect of longitudinal diffusion $D(\partial^2 c/\partial x^2)$ in Eq. (1.1) could be neglected. This assumption was based on an analysis he made of the results obtained.

For the concentration, averaged over the tube cross-section

$$c_0(x,t)=\langle c\rangle=\frac{2}{a^2}\int_0^a c(x,r,t)r\,dr \tag{1.4}$$

(the angular brackets here and henceforth denote quantities averaged over the tube cross-section) the following "conservation law" is obtained from (1.2)

$$\frac{\partial c_0}{\partial t}+\frac{\partial\langle J\rangle}{\partial x}=0 \tag{1.5}$$

where the average flow is

$$\langle J\rangle=\langle uc\rangle \tag{1.6}$$

In discussing this convection-diffusion problem Taylor notes that for practical purposes it is of interest to obtain a closed equation for the mean concentration c_0, i.e. to obtain some approximate expression for the mean flow $\langle J\rangle$ in terms of c_0. Starting from physical considerations (which he himself called intuitive), Taylor derived the following approximate equation

$$\frac{\partial c_0}{\partial t}+\langle u\rangle\frac{\partial c_0}{\partial x}=D_{\text{eff}}\frac{\partial^2 c_0}{\partial x^2} \tag{1.7}$$

where, for a Poiseuille laminar velocity profile

$$u=u_m[1-(r/a)^2]$$

and the effective diffusion coefficient equals

$$D_{\text{eff}}=a^2u_m^2/(192D)$$

It can be seen that the effective longitudinal dispersion is inversely proportional to the coefficient D (D = const).

In papers devoted to a rigorous derivation of Taylor's model and analogues for other problems (see the review in [3]), an attempt was made to justify or refine Taylor's assumptions, but no appreciable understanding is achieved in them as, for example, in [4], or they even contain errors (for example, in [3] the approximate expression for the concentration c violates the condition $\langle c\rangle = c_0$). A clear description of Taylor's results within the framework of his idea has been given by Levich [5].

The "closure problem" formulated by Taylor is a standard one in theoretical and mathematical physics. For example, in the language of the kinetic theory of gases Eq. (1.2) can be regarded as a "kinetic equation" for the distribution function $c(x, r, t)$, Eq. (1.5) can be regarded as the "conservation law" for its first (zeroth) moment, while Eq. (1.7) can be regarded as the "approximate hydrodynamic equation". The problem of deriving Eq. (1.7) from (1.2) itself has now become the problem of changing from a detailed "microscopic" description at the level of the "distribution function" $c(x, r, t)$ to a rough "macroscopic" description of the system using the hydrodynamic parameter $c_0(x, t)$, obtained from $c(x, r, t)$ using the averaging procedure.

In the kinetic theory of gases the corresponding problem is solved using the Chapman–Enskog method [6], which in the jargon of specialists is called the "expansion in gradients of the parameters of the hydrodynamic state". A general approach to solving this kind of problem has been given by Bogolyubov [7]. He notes that the problem of an abbreviated description of the systems is related to the problem of averaging non-linear dynamical systems over fast variables, for the solution of which he developed, together with Krylov, appropriate asymptotic methods. In the case considered here the fast variable is the transverse coordinate r, along which averaging is carried out.

Following Bogolyubov [6, Section 9], in order to emphasize in explicit form the fact that x is a slow variable and to avoid superfluous coordinate transformations, we will introduce a small parameter ε in a formal way and we will consider the distribution $c(\varepsilon, r, t)$ and construct asymptotic expansions in powers of this parameter. The parameter ε will be assumed to be equal to unity in the final formulae.

As in [6], since we wish to obtain an equation to study the "slow" process of the evolution of the hydrodynamic parameter c_0, we will construct a solution of dynamic equation (1.2) which depends functionally on time via the time-dependence of the "hydrodynamic parameter" $c_0(\xi, t)$ with $c\ \xi = \varepsilon x$ [7]

$$c(\xi, r, t) = c_0(\xi, t) + \sum_{k=1}^{\infty} \varepsilon^k c_k(\xi, r \mid c_0), \quad \langle c_k \rangle = 0 \tag{1.8}$$

$$\frac{\partial c_0}{\partial t} = -\varepsilon \frac{\partial \langle J \rangle}{\partial \xi} = -\varepsilon \sum_{k=0}^{\infty} \varepsilon^k \frac{\partial J_k}{\partial \xi}, \quad J_k = \langle u c_k \rangle \tag{1.9}$$

The functional time dependence through c_0 of the terms of expansion (1.8) implies that, when calculating their derivatives with respect to t we must differentiate the functions c_0 occurring in them and replace $\partial c_0/\partial t$ in accordance with expansions (1.9). In non-linear mechanics this procedure corresponds to the elimination of the secular terms, whereas in the kinetic theory and in the case considered it enables us to obtain an asymptotic description of the evolution "for long times" or "at the hydrodynamic level" [8].

In view of the linearity and structure of Eqs (1.8) and (1.9) it can be shown that the terms of the expansion c_k has the following structure

$$c_k = A_k(r) \frac{\partial^{(k)} c_0}{\partial \xi^k} \tag{1.10}$$

and, correspondingly,

$$J_k = \langle u A_k \rangle \frac{\partial^{(k)} c_0}{\partial \xi^k} \tag{1.11}$$

These formulae can also be used for $k = 0$ if we assume that the zeroth-order derivative is simply the operation of multiplication by unity. It is obvious that $A_0 = 1$. The functions $A_k(r)$ satisfy the equations which follow from Eq. (1.2) on substituting relations (1.8)–(1.11) and equating to zero the groups of terms of like powers of the parameter ε (which is equivalent to equating to zero terms with derivatives $\partial^{(k)} c_0/\partial \xi^k$ of the same order $k > 1$)

$$\frac{1}{r} \frac{d}{dr} r \frac{dA_{k+1}}{dr} = G_{k+1} \tag{1.12}$$

where

$$G_{k+1} = uA_k - \sum_{p=0}^{k} \langle u A_p \rangle A_{k-p}$$

The solution of Eq. (1.12) which satisfies boundary condition (1.3) can be written in the explicit form

$$A_{k+1}(r) = \int_0^r \frac{1}{r} \left(\int_0^r r G_{k+1} dr \right) dr + B_{k+1}$$

where the constant B_{k+1} is found from the condition $c_0 = \langle c \rangle$ or $\langle A_{k+1} \rangle = 0$ which guarantees that the corrections c_{k+1} make no contribution to the mean concentration c_0. Hence, all the terms of expansions (1.8) and (1.9) are defined uniquely (in explicit form).

We will present the results of calculations of the first few terms of the asymptotic series

$$A_0 = 1, \quad J_0 = \frac{u_m}{2}\frac{\partial c_0}{\partial x} \tag{1.13}$$

$$A_1 = -\frac{a^2 u_m}{16D}\left[\left(\frac{r}{a}\right)^4 - 2\left(\frac{r}{a}\right)^2 + \frac{2}{3}\right], \quad J_1 = -D_{eff}\frac{\partial c_0}{\partial x}, \quad D_{eff} = \frac{a^2 u_m^2}{192D}$$

In this order of the asymptotic theory the corresponding "hydrodynamic equation", i.e. the approximate equation for the mean concentration c_0, is exactly identical with Taylor's equation (1.7), which is now obtained as the product of the regular asymptotic procedure and can be refined by taking the following approximations into account. For example, in the following approximation (which are not henceforth used, the lengthy calculations and expressions being omitted)

$$J_2 = D_2\frac{\partial^2 c_0}{\partial x^2}, \quad D_2 = \frac{u_m^3 a^4}{23040D^2}$$

and the generalized dispersion equation for c_0 takes the form

$$\frac{\partial c_0}{\partial t} + \frac{u_m}{2}\frac{\partial c_0}{\partial x} = D_{eff}\frac{\partial^2 c_0}{\partial x^2} - D_2\frac{\partial^3 c_0}{\partial x^3} \tag{1.14}$$

The Chapman–Enskog method is often called the expansion in gradients of the parameters of the hydrodynamic state, since each following term of this expansion of the solution of Boltzmann's equation (for a low Knudsen number) contains a number of derivatives with respect to the spatial variable that is one greater than the previous term. In exactly the same way, the asymptotic expansions (1.8) and (1.9) contain in each subsequent term a derivative with respect to the longitudinal coordinate that is one higher than in the previous term, as is clearly shown in relations (1.10) and (1.11).

In dimensionless variables the series in formulae (1.8) and (1.9) will be expansions in powers of the dimensionless parameter $\varepsilon_* = a^2 u_m/(Dl)$, where l is the characteristic scale of change of c along the longitudinal coordinate.

Analysing the conditions of applicability of model (1.7), Taylor requires that the following order relations must be satisfied [1, 5]

$$D_{\text{eff}} \gg D, \qquad \varepsilon_* \ll 1$$

The first relation enables us to neglect the longitudinal molecular diffusion term on changing from Eq.(1.1) to (1.2), whereas the second can be rewritten in the form of the condition for the ratio of the transverse scale a to the longitudinal scale l to be small in the form

$$a/l \ll D/(au_m) \tag{1.15}$$

The following limit follows from the condition $D_{\text{eff}} \gg D$

$$D/(au_m) \ll 0.07 \tag{1.16}$$

The explicit expression for the correction of the next order to the Taylor expression obtained above enables us to relax condition (1.15) considerably. Really, for the Taylor model to be valid it is sufficient for the following condition to be satisfied

$$D_{\text{eff}}\,|\partial^2 c_0/\partial x^2| \gg D_2\,|\partial^3 c_0/\partial x^3|$$

or

$$a/l \ll 120(D/(au_m)) \tag{1.17}$$

which contains a numerical factor which relaxes requirement (1.15) by more than two orders of magnitude. Note that the condition $l \gg 0.12a$ follows from limits (1.15) and (1.16), but it is not sufficient for Taylor's model to be applicable if condition (1.17) is not satisfied.

The fundamental solution, called the source function or Green's function, of Eq. (1.7) for the main approximation, representing Taylor's model of longitudinal dispersion of a passive impurity in the case of the combined effect of transverse diffusion and convection in a non-uniform longitudinal velocity field, has the form

$$G(x,t) = (4\pi D_{\text{eff}}t)^{-1/2} \exp[-(x - \langle u \rangle t)^2 / (4D_{\text{eff}}t)]$$

$$\lim_{t\to 0} G(x,t) = \delta(x), \quad \int_{-\infty}^{\infty} G(x,t)dx = 1 \tag{1.18}$$

2. THE HYPERBOLIC MODEL OF THE DISPERSION OF A PASSIVE IMPURITY

In experiments on equipment constructed in accordance with the conditions of applicability of Taylor's theory [1, 2] it was confirmed that an asymptotic dispersion mode in the form (1.18) is established. Nevertheless, the description of unsteady dispersion based on model (1.7) has a defect, known as the paradox of infinite perturbation propagation velocity. It can be seen that, according to solution (1.18), a perturbation produced at the instant of time $t = 0$ at the point $x = 0$ will, at any subsequent instant of time $t > 0$, be perceived at any distance from the point where it arises, i.e. the velocity of propagation of the perturbations is infinite. At the same time it is obvious that Eq.(1.2) does not possess this property: the maximum velocity with which a tracing particle can propagate in the longitudinal direction cannot exceed the maximum flow velocity u_m. This paradox was solved in [9] by adding an "artificial propagation term" to the diffusion equation and converting it into a hyperbolic-type telegraph equation, which ensures that the perturbation propagation velocity is finite and that the solution has a diffusion form when this velocity approaches infinity.

Similar problems arise in many areas of mathematical and theoretical physics (see, for example, the reviews [10, 11] in which more than two hundred publications are cited on this question in heat-conduction problems alone) and, in particular, in the classical hydrodynamics of a viscous heat-conducting medium [12]. The artificial addition of the second derivative with respect to time in the parabolic diffusion and heat-conduction equations or additional terms of a relaxation type with time derivatives in Fick's and Fourier's transport laws, relating mass and heat flows to the concentration and temperature gradients they generate, does not solve the problem, since it is theoretically necessary to calculate the values of the coefficients of these terms using the initial model of the phenomenon. This problem has been solved for the cases of a single-component medium [12–14] and a multicomponent medium [15–17] using generalized normal solutions of Boltzmann's equation [14, 18, 19]. For the problem in question of unsteady longitudinal dispersion of a passive impurity in a steady flow of fluid in a cylindrical tube, wave (i.e. hyperbolic) models have also been developed [20–24].

Below, we will use the Bubnov–Galerkin method to construct a hyperbolic model of the longitudinal dispersion of a passive impurity, but, unlike [20, 21, 25], where it is also employed, here the test functions and approximate solutions will be constructed using the results of the asymptotic theory developed in Section 1. As was further shown, this ensures that the hyperbolic models are matched to the results of the asymptotic method under conditions when the latter is applicable.

In dimensionless variables

$$\tau = Dt/a^2, \quad \rho = r/a, \quad \xi = Dx/(a^2u_m)$$

the equation of unsteady convection-diffusion of a passive impurity in steady laminar flow in a cylindrical tube (1.2) takes the form

$$Lc = 0 \tag{2.1}$$

$$L = \frac{\partial}{\partial t} + \upsilon \frac{\partial}{\partial \xi} - \Delta, \quad \upsilon = 1 - \rho^2, \quad \Delta c = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial c}{\partial \rho}\right)$$

The impurity concentration c is subject to the boundary condition

$$\partial c/\partial \rho = 0 \quad \text{when} \quad \rho = 1 \tag{2.2}$$

The initial condition is specified in time depending on the specific problem and is omitted here.

To derive the hyperbolic model of unsteady dispersion we will use the ideas of the Bubnov–Galerkin method and the results of the asymptotic analysis of the problem given above.

According to this method, the approximate solution of Eq.(2.1) can be represented in the form of a section of a series

$$c_N(\xi,\rho,\tau) = c_0(\xi,\tau) + \sum_{k=1}^{N} a_k(\xi,\tau)\varphi_k(\rho) \tag{2.3}$$

Here the test functions $\varphi_k(\rho)$ need not be normalized or mutually orthogonal but must satisfy the conditions of linear independence: no function φ_k can be represented in the form of a linear combination of other test functions. Moreover, the functions φ_k are subject to the natural conditions

$$\partial\varphi_k/\partial\rho = 0 \text{ when } \rho = 1 \text{ and } \rho = 0 \tag{2.4}$$

necessary to satisfy conditions (2.2) and (1.4). Here the angular brackets now mean an averaging operation of the form

$$\langle F(\rho)\rangle = 2\int_0^1 F(\rho)\rho d\rho$$

To obtain the equations from which the coefficients c_0 and a_k of (2.3) are determined, we can use various approaches, particularly for small N, but the classical approach is the most natural one. According to this these equations are determined from the system of equations

$$\langle\varphi_k(Lc_N)\rangle = 0, \quad k = 0,1,\dots,N; \quad \varphi_0 = 1$$

where L is the operator of Eq.(2.1). In explicit form, the equations for determining the coefficient functions c_0 and a_k have the form

$$\frac{\partial c_0}{\partial\tau} + \langle v\rangle\frac{\partial c_0}{\partial\xi} + \sum_{k=1}^{N} V_k\frac{\partial a_k}{\partial\xi} = 0, \quad V_k = \langle v\,\varphi_k\rangle$$

$$\sum_{k=1}^{N} A_{kp}\frac{\partial a_k}{\partial\tau} + \sum_{k=0}^{N} V_{kp}\frac{\partial a_k}{\partial\xi} - \sum_{k=1}^{N}\Delta_{kp}a_k = 0, \quad p = 1,\dots,N$$

$$A_{kp} = \langle\varphi_k\varphi_p\rangle, \quad V_{kp} = \langle v\,\varphi_k\varphi_p\rangle, \quad \Delta_{kp} = \langle\varphi_p\langle\Delta\varphi_k\rangle\rangle$$

Various systems of test functions φ_k are considered in the publications mentioned above. Below, we propose to use as the functions φ_k the polynomials $A_k(\rho)$, reduced to dimensionless form, which satisfy condition (2.4).

We will construct the simplest hyperbolic model of longitudinal dispersion using the two-term Bubnov-Galerkin representation

$$c(\rho,\xi,\tau) = c_0(\xi,\tau) + a_1(\xi,\tau)\varphi_1(\rho)$$

where, according to the results (1.13) obtained above, we can choose as the function φ_1

$$\varphi_1(\rho) = -(\rho^4 - 2\rho^2 + 2/3)/16$$

Hence, we obtain a system of two equations for the functions c_0 and a_1

$$\frac{\partial c_0}{\partial\tau} + \frac{1}{2}\frac{\partial c_0}{\partial\xi} - \frac{1}{192}\frac{\partial a_1}{\partial\xi} = 0 \tag{2.5}$$

$$\frac{\partial a_1}{\partial\tau} - 15\frac{\partial c_0}{\partial\xi} + \frac{5}{8}\frac{\partial a_1}{\partial\xi} + 15a_1 = 0$$

At first glance it would seem that the two-term Bubnov–Galerkin approximation is an extremely simplified description of unsteady dispersion of a passive impurity. However, in fact, it contains not only the Taylor model, characterized by an effective diffusion coefficient, but also a term of the next order of the asymptotic theory with an exact value of the coefficient D_2.

This can be shown by two methods.

First method. We write the second equation of system (2.5) in the form

$$a_1 = \frac{\partial c_0}{\partial \xi} - \frac{1}{120}\frac{\partial a_1}{\partial \xi} - \frac{1}{15}\left(\frac{\partial a_1}{\partial \tau} + \frac{1}{2}\frac{\partial a_1}{\partial \xi}\right)$$

We recall that, in the asymptotic theory, the coefficient a_1 represents the corrections to c_0, i.e. it is a term of the next approximation with respect to c_0, and, moreover, the scale along the longitudinal coordinate is assumed to be large while the corresponding derivative $\partial/\partial\xi$ is assumed to be small. Hence, the relation written above can be integrated by substituting into the right-hand side, instead of the quantity a_1, its principal value $\partial c_0/\partial\xi$ and we thus obtain

$$a_1 \approx \frac{\partial c_0}{\partial \xi} - \frac{1}{120}\frac{\partial^2 c_0}{\partial \xi^2} - \frac{1}{2880}\frac{\partial^2 a_1}{\partial \xi^2}$$

The last term in this relation has a higher order, the third ($O(\partial^3 c_0/\partial\xi^3)$), than the first two terms (first- and second-order infinitesimals), and it can be dropped. Substituting the expression obtained for a_1 into the first equation of system (2.5), we obtain

$$\frac{\partial c_0}{\partial \tau} + \langle \upsilon \rangle \frac{\partial c_0}{\partial \xi} = \frac{1}{192}\frac{\partial^2 c_0}{\partial \xi^2} - \frac{1}{23040}\frac{\partial^3 c_0}{\partial \xi^3}$$

It can be seen that this equation agrees completely with Eq.(1.14) for c_0, if we rewrite the latter in dimensionless form.

The second method. The second method employed when analysing linear systems consists of comparing the dispersion relations for the Taylor and hyperbolic models of the longitudinal dispersion of a passive impurity.

To simplify the later calculations we will change to a frame of reference moving with the average velocity of the carrying liquid $\langle \upsilon \rangle = 1/2$. Assuming $\tau' = \tau$, $\eta = \xi - \langle \upsilon \rangle \tau$, instead of (2.5) we have

$$\begin{aligned} &\frac{\partial c_0}{\partial \tau'} = \frac{1}{192}\frac{\partial a_1}{\partial \eta} \\ &\frac{\partial a_1}{\partial \tau'} + \frac{1}{8}\frac{\partial a_1}{\partial \eta} - 15\frac{\partial c_0}{\partial \eta} + 15 a_1 = 0 \end{aligned} \tag{2.6}$$

Consider the dispersion relation for system of equations (2.6), corresponding to its solutions with a space–time dependence of the form $\exp(i\omega\tau' - ik\eta)$. It can be reduced to a quadratic equation, the roots of which are

$$\omega_\pm(k) = \frac{1}{2}\left(\frac{k}{8} + 15i\right)\left\{1 \pm \left[1 + \frac{5}{16}k^2\left(\frac{k}{8} + 15i\right)^{-1}\right]^{1/2}\right\}$$

Since each power of $-ik$ appears as a result of differentiating the solution with respect to the spatial variable (or ξ), the expansion in power series in $-ik$ corresponds to the asymptotic Taylor expansion (1.9). These expansions of the roots of the dispersion relation have the form

$$i\omega_-(k) = \frac{(-ik)^2}{192} - \frac{(ik)^3}{23040} + O(k^4)$$

$$i\omega_+(k) = -15 + \frac{ik}{8} - i\omega_-(k)$$

The root $\omega_-(k)$ corresponds to a diffusion-type mode and agrees with Eq.(1.14) for c_0 with correct values of the first two coefficients on the right-hand side, whereas the root $\omega_-(k)$ corresponds to a rapidly propagating attenuating mode. Hence, an analysis of the dispersion relation also shows that the two-term approximation, being represented in the form of an expansion in gradients (i.e. in powers of $-ik$), agrees, apart from third-order terms, with the Taylor asymptotic expansion.

System (2.26) reduces to a single second-order hyperbolic-type equation

$$\frac{\partial^2 c_0}{\partial \tau'^2}+\frac{1}{8}\frac{\partial^2 c_0}{\partial \tau' \partial \eta}-\frac{5}{64}\frac{\partial^2 c_0}{\partial \eta^2}+15\frac{\partial c_0}{\partial \tau'}=0 \tag{2.7}$$

To eliminate the cross derivative from Eq. (2.7) we make a replacement of variables, putting

$$\vartheta=\tau',\ \zeta=\eta-\frac{\tau'}{16},\ \ \frac{\partial}{\partial \tau'}=\frac{\partial}{\partial \vartheta}-\frac{1}{16}\frac{\partial}{\partial \zeta},\ \ \frac{\partial}{\partial \eta}=\frac{\partial}{\partial \zeta}$$

We then reduce the equation obtained to the telegraph equation by replacing the dependent variable

$$c_0(\vartheta,\zeta)=w(\vartheta,\zeta)\exp(\alpha\vartheta+\beta\zeta)$$

where α and β are parameters. We obtain the following equation for the function w

$$\frac{\partial^2 w}{\partial \vartheta^2}-v_*^2\frac{\partial^2 w}{\partial \zeta^2}+(2\alpha+15)\frac{\partial w}{\partial \vartheta}+\left(-2\beta v_*^2-\frac{15}{16}\right)\frac{\partial w}{\partial \zeta}+$$
$$+\left(\alpha^2-\beta^2 v_*^2+15\alpha-\frac{15}{16}\beta\right)w=0$$

where $v^*=\sqrt{21}/16$ is the phase velocity of propagation of the perturbation fronts to the right and to the left from the source in a frame of reference moving relative to the initial frame of reference with velocity

$$v_r=9/16 \tag{2.8}$$

The parameters α and β are chosen so that the coefficients of terms containing w and $\partial w/\partial \xi$ vanish, while the coefficient of $\partial w/\partial \vartheta$ is positive (the last condition serves for choosing the required root of the quadratic equation for the parameter α and ensures agreement between the solution in the asymptotic region and the solution obtained in Section 1)

$$\beta=-\frac{15}{32 v_*^2},\ \ \alpha=-\frac{15}{2}+\frac{15}{2}\sqrt{1-\delta},\ \ \delta=\frac{1}{(16 v_*)^2}$$

The function w now satisfies the telegraph equation

$$\frac{\partial^2 w}{\partial \zeta^2}-\frac{1}{v_*^2}\frac{\partial^2 w}{\partial \vartheta^2}-\mu^2\frac{\partial w}{\partial \vartheta}=0,\ \ \mu^2=15\frac{\sqrt{1-\delta}}{v_*^2} \tag{2.9}$$

The solution of Cauchy's problem for Eq.(2.9) with initial data

$$w=w_0(\zeta),\ \ \partial w/\partial\vartheta=w_1(\zeta)\ \text{ when }\ \vartheta=0 \tag{2.10}$$

can be written, using Green's function [9] with shifted arguments $G(\zeta-\zeta',\ \vartheta-\vartheta')$, in the form

$$w(\zeta,\vartheta)=\mu^2\int_{-\infty}^{\infty}[w(\zeta',\vartheta')G(\zeta-\zeta',\vartheta-\vartheta')]_{\vartheta'=0}d\zeta'+$$
$$+\frac{1}{v_*^2}\int_{-\infty}^{\infty}\left[G(\zeta-\zeta',\vartheta-\vartheta')\frac{\partial w(\zeta',\vartheta')}{\partial\vartheta'}-w(\zeta',\vartheta')\frac{\partial G(\zeta-\zeta',\vartheta-\vartheta')}{\partial\vartheta'}\right]_{\vartheta'=0}d\zeta' \tag{2.11}$$

When constructing Green's function we first note that the initial equation (2.9) allows of a regular particular solution of the form

$$w_p(\zeta,\vartheta)=\exp\left(-\tfrac{1}{2}\mu^2 \upsilon_*^2\vartheta\right)I_0\left(\tfrac{1}{2}\mu^2\upsilon_*\sqrt{\upsilon_*^2\vartheta^2-\zeta^2}\right) \quad (2.12)$$

where $I_0(z)$ is the solution of the modified Bessel equation of zeroth order

$$I_0''+z^{-1}I_0'-I_0=0$$

which is subject to the initial conditions

$$I_0(0)=1,\quad I_0'(0)=0$$

it is now easy to show that

$$G(\zeta,\vartheta)=\tfrac{1}{2}\upsilon_* w_p(\xi,\vartheta)H(\upsilon_*\vartheta-|\zeta|) \quad (2.13)$$

where $H(z)$ is the Heaviside unit step function, which is equal to zero for negative values of the argument, and equal to unity for positive values of the argument.

Substituting relations (2.10) and (2.13) into (2.11), we obtain an explicit expression for the solution of Cauchy problem (2.9), (2.10) in terms of the particular solution (2.12) and the initial data

$$\begin{aligned} w(\zeta,\vartheta)&=\tfrac{1}{2}\exp\left(-\tfrac{1}{2}\mu^2\upsilon_*^2\vartheta\right)[w_0(\zeta+\upsilon_*\vartheta)+w_0(\zeta-\upsilon_*\vartheta)]+\\ &+\frac{1}{2\upsilon_*}\int_{\zeta-\upsilon_*\vartheta}^{\zeta+\upsilon_*\vartheta}\left\{\exp(-\mu^2\upsilon_*^2\vartheta)\frac{\partial w_p(\zeta-\zeta',-\vartheta)}{\partial\vartheta}w_0(\zeta')+\right.\\ &\left.+\ w_p(\zeta-\zeta',\vartheta)w_1(\zeta')\right\}d\zeta' \end{aligned} \quad (2.14)$$

Investigating the behaviour of the terms contained in the solution as $\mu\to 0$, we obtain that

$$w(\zeta,\vartheta)=\frac{1}{2}[w_0(\zeta+\upsilon_*\vartheta)+w_0(\zeta-\upsilon_*\vartheta)]+\frac{1}{2\upsilon_*}\int_{\zeta-\upsilon_*\vartheta}^{\zeta+\upsilon_*\vartheta}w_1(\zeta')d\zeta'+O(\mu^2)$$

Hence, as $\mu\to 0$, formula (2.14) becomes d'Alembert's solution of Cauchy's problem for the one-dimensional wave equation.

We now note that the presence in Green's function of a factor in the form of the Heaviside function $H(\upsilon_*\vartheta-|\zeta|)$, which is equal to zero when $|\zeta|>\upsilon_*\vartheta$, confirms that the propagation velocity of the perturbations is finite and also confirms the presence of perturbation jumps or fronts when $\zeta=\pm\upsilon_*\vartheta$. The behaviour of Green's function (2.13) far from the fronts, i.e. when $|\zeta|/(\upsilon_*\vartheta)\ll 1$ can be obtained by formally assuming $\upsilon_*\to\infty$ and using the asymptotic form of the Bessel function $I_0(z)$ for large values of the argument

$$I_0(z)=(2\pi t)^{-1/2}e^z[1+O(z^{-1})]$$

Consequently,

$$G(\zeta,\vartheta)=\frac{1}{2\sqrt{\pi\mu^2\vartheta}}\exp\left(-\frac{\mu^2\zeta^2}{4\vartheta}\right)H(\vartheta)[1+O(\upsilon_*^{-2})]$$

i.e. the principal term of the asymptotic form of Green's function as $\upsilon_*\to\infty$ is identical with Green's function for the diffusion equation.

We will also consider the asymptotic form of solution (2.14) as $\upsilon_*\to\infty$. Since

$$\frac{1}{2\upsilon_*}\exp\left(-\mu^2\upsilon_*^2\vartheta\right)\frac{\partial w_p(\zeta-\zeta',-\vartheta)}{\partial\vartheta}=$$

$$= \frac{\mu}{2\sqrt{\pi\vartheta}} \exp\left[\frac{-\mu^2(\zeta-\zeta')^2}{4\vartheta}\right] [1 + O(\upsilon_*^{-2})]$$

then, apart from terms $O(\upsilon_*^{-2})$, solution (2.14) can be represented in the form

$$w(\zeta,\vartheta) = \frac{\mu}{2\sqrt{\pi\vartheta}} \int_{-\infty}^{\infty} \exp\left[-\frac{\mu^2(\zeta-\zeta')^2}{4\vartheta}\right] w_0(\zeta')d\zeta'$$

which corresponds to the solution of Cauchy's problem for the diffusion equation

$$\frac{\partial^2 w}{\partial \zeta^2} - \mu^2 \frac{\partial w}{\partial \vartheta} = 0, \quad w(\zeta,0) = w_0(\zeta)$$

We will consider the formal problem of a pulsed δ-shaped source for Eq.(2.9), assuming

$$w_0(\zeta) = \delta(\zeta), \quad w_1(\zeta) = 0$$

From the general formula (2.14) of the solution of Cauchy's problem for these initial data we obtain

$$w(\zeta,\vartheta) = \frac{1}{2}\exp\left(-\frac{1}{2}\mu^2\upsilon_*^2\vartheta\right) [\delta(\zeta+\upsilon_*\vartheta) + \delta(\zeta-\upsilon_*\vartheta)] + \\ + \frac{1}{2\upsilon_*}\exp(-\mu^2\upsilon_*^2\vartheta)H(\upsilon_*\vartheta - |\zeta|)w_p(\zeta,-\vartheta) \tag{2.15}$$

The first term in (2.15) is analogous to d'Alembert's solution, which in this case again contains an exponentially attenuating factor. The second term, as can easily be shown, using the asymptotic estimates derived above, as $\upsilon_* \to \infty$ gives the source function for the diffusion equation.

The source function (2.15) is non-zero in the region $|\zeta| \leqslant \upsilon_*\vartheta$, i.e. a pulsed source which began to act at the instant of time $t = 0$, generates, when $t > 0$, a perturbed region of finite size, the boundaries of which propagate to the right and left with velocities $\pm\upsilon_*$. We also recall that the coordinate ζ is measured in a frame of reference moving with respect to the initial system, connected with the tube, with a velocity υ_r (2.8). Hence, in the initial frame of reference the fronts move in the tube with velocities $\upsilon_r + \upsilon_*$, equal to $\upsilon_1 = 0.8489$ and $\upsilon_2 = 0.2761$, respectively (we recall that these dimensionless velocities are measured in units of u_m – the maximum velocity at the centre of the tube).

It must, however, be borne in mind that these conclusions are obtained on the basis of the approximate mathematical one-dimensional model of the phenomenon, which, unlike the Taylor model of longitudinal dispersion, takes into account the initial stage of the development of the process approximately. In view of the approximate nature of the model obtained we can only assume that the fundamental mechanisms and features of the dispersion of passive impurity – the interaction between transverse diffusion and convection in a non-uniform velocity field and with a finite propagation velocity of the perturbations – will be adequately described by system of equations (2.5) or Eq.(2.7), whereas other effects, for example, the effect of molecular diffusion in a longitudinal direction, can be neglected.

We will now write the equations of the hyperbolic one-dimensional model of dispersion in dimensional form. To do this we will introduce the dimensional concentration flux in the system, moving with an average velocity of the fluid $\langle\mu\rangle$

$$j = \langle(u - \langle u\rangle)c\rangle \tag{2.16}$$

In the approximation considered

$$j = a_1 u_m \langle \upsilon \varphi_1\rangle = -a_1 u_m/192 \tag{2.17}$$

Reverting in Eqs (2.5) to dimensional variables and expressing a_1 in terms of j, by (2.17) we have

$$\begin{aligned} &\frac{\partial c_0}{\partial t}+\frac{u_m}{2}\frac{\partial c_0}{\partial x}+\frac{\partial j}{\partial x}=0 \\ &\frac{\partial j}{\partial t}+\frac{5}{8}u_m\frac{\partial j}{\partial x}+\frac{5}{64}u_m^2\frac{\partial c_0}{\partial x}+\frac{15D}{a^2}j=0 \end{aligned} \qquad (2.18)$$

Eliminating the variable j from system (2.18), we arrive at the following hyperbolic equation for c_0

$$\frac{\partial c_0^2}{\partial t^2}+\frac{9}{8}u_m\frac{\partial^2 c_0}{\partial t\partial x}+\frac{15}{64}u_m^2\frac{\partial^2 c_0}{\partial x^2}+\frac{15D}{a^2}\left(\frac{\partial c_0}{\partial t}+\frac{u_m}{2}\frac{\partial c_0}{\partial x}\right)=0$$

In conclusion we note that the theory described can be extended to other cases of longitudinal dispersion [26], which differ geometrically from the simple case considered here (non-circular cross-sections, flows in ring gaps, three-dimensional flows with predominant longitudinal convection, etc.), the state of the moving medium (turbulent flow and flows of non-Newtonian fluids) or by the effect of physical-chemical transitions.

REFERENCES

1. TAYLOR, G., Dispersion of soluble matter in solvent flowing slowly through a tube. *Proc. Roy. Soc. London. Ser. A*, 1953, **219**, 1137, 186–203.
2. TAYLOR, G., The dispersion of matter in turbulent flow through a pipe. *Proc. Roy Soc. London. Ser. A*, 1954, **223**, 1155, 446–468.
3. FISCHER, H. B., Longitudinal dispersion and turbulent mixing in open-channel flow. *Ann. Rev. Fluid Mech.*, 1973, **5**, 59–73.
4. ELDER, J. W., The dispersion of marked fluid in turbulent shear flow. *J. Fluid Mech.*, 1959, **5**, Pt 4, 544–560.
5. LEVICH, V. G., *Physico chemical Hydrodynamics*. Prectice-Hall, Englewood Cliffs, NJ, 196.
6. CHAPMAN, S. and COWLING, T. G., *The Mathematical Theory of Non-Uniform Gases*. Cambridge University Press, Cambridge, 1952.
7. BOGOLYUBOV, N. N., *Dynamic Theory in Statistical Physics*. Gostekhizdat, Moscow, 1946.
8. KHON'KIN, A. D., Analysis of the convergence of the asymptotic solutions of kinetic theory using a model kinetic equation having an exact solution. *Teor. Mat. Fiz.*, 1991, **86**, 1, 130–135.
9. MORSE, Ph.M. and FESHBACH H., *Methods of Theoretical Physics*, Part 1, McGraw-Hill, London, 1953.
10. JOSEPH, D. D. and PREZIOSI L., Heat waves. *Rev. Mod. Phys.*, 1989, **61**, 1, 41–73.
11. JOSEPH, D. D. and PREZIOSI, L., Addendum to the paper Heat waves. *Rev. Mod. Phys.*, 1990, **62**, 2, 375–391.
12. KHON'KIN, A. D., The paradox of the infinite propagation velocity perturbations in the hydrodynamics of a viscous heat-conducting medium and in the equations of the hydrodynamics of fast processes. In *Aerodynamics*. Nauka, Moscow, 1976.
13. KHON'KIN, A. D., The equations of the hydrodynamics of fast processes. *Dokl. Akad. Nauk SSSR*, 1973, **210**, 5, 1033–1035.
14. KHON'KIN, A. D., Asymptotic methods in the theory of Boltzmann's equation and the equations of hydrodynamics. *Trudy TsAGI*, 1977, 1894, 3–27.
15. BALABANYAN, G. O. and KHON'KIN, A. D., The construction of generalized normal solutions of the kinetic equations of gas mixtures. *Teor. Mat. Fiz.*, 1974, **18**, 1, 130–137.
16. KHON'KIN, A. D., The hydrodynamic equations of a gas mixture with finite propagation velocities of perturbations. *Modelirovaniye v Mekhanike*, 1988, **2(19)**, 5, 142–147.
17. KHON'KIN, A. D. and ORLOV, A. V., Derivation of the modified diffusion equations in a gas mixture. *Phys. Rev.*, 1994, Ser. E., **49**, 1, 906–909.
18. ZUBAREV, D. N. and KHON'KIN, A. D., A method of constructing normal solutions of the kinetic equations using the boundary conditions. *Teor. Mat. Fiz.*, 1972, **11**, 3, 403–412.
19. KHON'KIN, A. D., Projection – asymptotic methods of solving Boltzmann's equation. In *Numerical Methods of Continuum Mechanics*. Inst. Teoreti. Prikl. Mekh. Sib. Otd. Akad. Nauk SSSR, 1978, **8**, 6, 132–141.
20. DIL'MAN V. V. and KRONBERG, A.YE., Relaxation phenomena in longitudinal mixing. *Teor. Osnovy Khim. Tekh.*, 1983, **17**, 5, 614–529.
21. DIL'MAN V. V. and KRONBERG, A.YE., The ratio of the time scales of a process and the modelling of chemical reactors. *Khim. Prom.*, 1983, 8, 16–21.
22. DIL'MAN V.V., An estimate of the longitudinal dispersion of material impurity in the turbulent motion of a fluid. *Khim. Prom.*, 1989, 8, 33–37.
23. WESTERTERP, K. R., DIL'MAN V. V., KRONBERG, A.YE. and BENNEKER, A., A wave model of longitudinal mixing. *Teor. Osnovy Khim Tekh.*, 1995, **29**, 6, 580–587.
24. WESTERTERP, K. R., DIL'MAN V. V. and KRONBERG, A.YE. Wave model for longitudinal dispersion: Development of model. *AIChE J.*, 1995, **41**, 9, 2013–2028.
25. WESTERTERP, K. R., DIL'MAN V. V., KRONBERG, A.YE. and BENNEKER, A.H., Wave model for longitudinal dispersion: Analysis and applications. *AIChE J.*, 1995, **41**, 9, 2029–2039.
26. DIL'MAN V. V. and POLYANIN, A. D., *Model Equation Methods and Analogies in Chemical Engineering*. Khimiya, Moscow, 1988.

Translated by R.C.G.